Midterm - Differential Equations (2018-19)

Time: 3 hours. Attempt all questions.

1. Let I be a time interval and let $u : I \to \mathbf{R}^+$ be a continuous nonnegative function obeying the inequality
$$u(t) \leq A + \epsilon F\big(u(t)\big)$$
for some $A, \epsilon > 0$ and some function $F : \mathbf{R}^+ \to \mathbf{R}^+$ which is locally bounded. Suppose also that $u(t_0) \leq 2A$ for some $t_0 \in I$. If ϵ is sufficiently small depending on A and F, show that in fact $u(t) \leq 2A$ for all $t \in I$. [**4 marks**]

2. Find the general solution of $u'' + 4u' + 4u = e^{-2x}$, and then find the particular solution with $u(0) = u'(0) = 0$. [**5 marks**]

3. let u_1 and u_2 be two solutions of the homogeneous equation $u'' + P(t)u' + Q(t)u = 0$ in an interval $[a, b]$, where P and Q are continuously differentiable functions. Suppose u_1 and u_2 have an extrema (maxima or minima) at the same point x_0 in the interval. Show that one of the solutions is a multiple of the other. [**4 marks**]

4. Find the general solution of $4xu'' + 2u' + u = 0$. [**5 marks**]

5. Let $q : \mathbf{R} \to \mathbf{R}$ be a continuously differentiable function such that $q(x) < 0$ for all x. If u is a *nontrivial* solution of $u'' + q(x)u = 0$, then show that u has at most one zero. [**4 marks**]

6. Find the general solution of
$$\begin{cases} \frac{dx}{dt} &= -3x + 4y \\ \frac{dy}{dt} &= -2x + 3y \end{cases} \qquad [\textbf{4 marks}]$$

7. Consider the nonlinear system
$$\begin{cases} \frac{dx}{dt} &= F(x, y) \\ \frac{dy}{dt} &= G(x, y) \end{cases}$$
where F and G are continuous and have continuous first partial derivatives. Assume that $(0, 0)$ is a critical point for the system. Show that if there exists a Liapunov function for the above system, then the critical point $(0, 0)$ is stable. [**4 marks**]